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Helical instabilities of slowly divergent jets

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The inviscid spatial growth of spiral modes of circular, slowly diverging jets is analysed. A multiple-scales expansion is used to develop a linear stability study for non-axisymmetric disturbances of arbitrary helicity. The numerical evaluation is restricted to axisymmetric modes and to the first two helical modes. It is shown that in the case of comparatively high values of the Strouhal number the modes exhibit a very rapid growth and reach their maximal amplification after a short distance, the axisymmetric instabilities being excited more strongly than their spiral counterparts. Contrary to this, the modes grow comparatively slowly in the case of smaller values of the Strouhal number and exhibit their maximal amplification further downstream. In the latter case the first spiral mode is more unstable than the axisymmetric one. A comparison with experiments seems to support these results.

1. Introduction

Mainly because of the investigation of jet noise, a great number of studies deal with the problem of jet instability and turbulence. Experimental research by Crow & Champagne (1971) showed that the natural frequency β_n gives the Strouhal number $St = \beta_n R/\pi U_0$ (R is the jet radius and U_0 the centre-line velocity) at the jet centre-line a value of 0·3. Using comparatively high forcing levels, they found that the axisymmetric mode reached a maximal amplification at the axial location $x = 11R$ and decayed further downstream. In order to ensure linearity, Moore (1977) performed experiments at much lower forcing levels and found that the fluctuating intensities exceeded those of Crow & Champagne. Experiments at various radial positions are reported by Chan (1977). In agreement with his theoretical considerations, he found the spiral modes (with helicity numbers m equal to one and two, respectively) for $St = 0{\cdot}5$ to be excited less strongly than the corresponding axisymmetric disturbances. Spectral measurements of different modes were performed by Armstrong (1977), who showed that the first spiral mode is the dominant instability in the centre of the boundary layer. In agreement with the results of Chan, exceptions to this were found in a comparatively narrow range of frequencies around $St = 0{\cdot}5$.

As regards theoretical investigations of jet instability, a quasi-parallel theory was formulated by Michalke (1971) and indicated that axisymmetric as well as spiral modes exhibit growth rates of equal order of magnitude. Using a multiple-scales method Bouthier (1972) developed a theoretical framework for dealing with slightly non-parallel flows. Later, Crighton & Gaster (1976) analysed spatial growth of axisymmetric disturbances in slowly diverging jets. They indicated that the growth rates and the phase speeds depend on the axial as well as on the radial positions and

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that they are different for the various disturbance quantities. By comparison with the experiments of Crow & Champagne and of Moore, they were able to support their theoretical findings.

The multiple-scales method yields a cumulative description of the effect of jet divergence and seems to be a more satisfying theoretical way to include the influence of jet spreading than the energy integral method proposed by Chan.

Because of the theoretical results of Michalke and the experimental results obtained by Armstrong, it is of interest to examine non-axisymmetric modes. We tackle this problem by an extension of the multiple-scales method and restrict ourselves to the axisymmetric modes and the first two helical modes.

2. Analysis

We use cylindrical co-ordinates (x, r, ϕ). The mean flow is given by

$$[U_x(X,r), \epsilon U_r(X,r), 0],$$

where $X = \epsilon x$ is a slow co-ordinate and ϵ is a small parameter characterizing the jet divergence. The linearized inviscid disturbance equations are

$$(\partial/\partial t + N)\, u'_x + N' U_x = -\partial p'/\partial x, \tag{1}$$

$$(\partial/\partial t + N)\, u'_r + \epsilon N' U_r = -\partial p'/\partial r, \tag{2}$$

$$(\partial/\partial t + N + \epsilon U_r/r)\, u'_\phi = -r^{-1}\partial p'/\partial \phi \tag{3}$$

($N = \epsilon U_r \partial/\partial r + U_x \partial/\partial x$ and $N' = u'_r \partial/\partial r + u'_x \partial/\partial x$) and a further equation is given by the familiar equation of continuity. According to Bouthier, we introduce a fast variable $s = g(X)/\epsilon$, which plays the same role for slow divergence as x does in parallel flow. Hence if we define $\mathbf{H} = (u'_x, u'_r, u'_\phi, p')$ then

$$\mathbf{H}(X, s, r, \phi, t) = [\mathbf{F}(X,r) + \epsilon \mathbf{G}(X,r) + \ldots] \exp\{i[g(X)/\epsilon + m\phi - \beta t]\}, \tag{4}$$

with $\mathbf{F} = (F_x, F_r, F_\phi, F_p)$ and $\mathbf{G} = (G_x, G_r, G_\phi, G_p)$. Insertion of (4) into (1)–(3) yields at leading order (ϵ^0)

$$i(aU_x - \beta) F_x + U'_x F_r = -iaF_p, \tag{5}$$

$$i(aU_x - \beta) F_r = -F'_p \tag{6}$$

and

$$i(aU_x - \beta) F_\phi = -imF_p/r, \tag{7}$$

where we have put $a = dg(X)/dX$ and use a prime to denote $\partial/\partial r$. Since X plays the role of a parameter in (5)–(7), these equations are those of a locally parallel flow and neglecting the higher correction in (4) we may write

$$\mathbf{H} = A(X)\,\mathbf{F}^0(X,r) \exp\left\{\frac{i}{\epsilon}\int_{X_0}^{X} a(X')\,dX' + i(m\phi - \beta t)\right\}, \tag{8}$$

where $\mathbf{F}^0$ denotes the solution vector of the locally parallel problem and $\mathbf{F}(X, r) = A(X) F^0(X, r)$. If we use the equation of continuity in (5)–(7) to eliminate all variables in favour of the pressure, we get

$$F_p^{0''} + [1/r - 2U'_x/(U_x - c)]\, F_p^{0'} - (m^2/r^2 + a^2)\, F_p^0 = 0 \quad (c = \beta/a). \tag{9}$$

Since U_x vanishes very rapidly as $r \to 0$ [cf. (24)] we get from (9) an eigenfunction finite at $r = 0$:

$$F_p^0 \to C I_m(ar) \quad \text{as} \quad r \to 0, \tag{10}$$

where I_m denotes the modified Bessel function of order m. The constant of integration C has to be fixed by a convenient normalization. While for $m = 0$ we use the same normalization as Crighton & Gaster, for $m = 1$ we demand $F_r^0(r=0) = 1$, which yields $C = 2i(c-1)$. Finally, we use $F_r^0 \to r$ as $r \to 0$ or $C = 4i(c-1)/a$ for $m = 2$. The adjoint eigenfunction is given by

$$\tilde{F}_p^0 = rF_p^0/(U_x - c)^2. \tag{11}$$

At order ϵ^1, (1)–(4) give

$$i(aU_x - \beta)\, G_x + G_r U_x' + iaG_p = -U_r F_x' - \partial(U_x F_x + F_p)/\partial x, \tag{12}$$

$$i(aU_x - \beta)\, G_r + G_p' = -U_x \, \partial F_r/\partial X - (U_r F_r)', \tag{13}$$

$$i(aU_x - \beta)\, G_\phi + imG_p/r = -U_x \, \partial F_\phi/\partial X - U_r (rF_\phi)'/r. \tag{14}$$

Finally, the equation of continuity leads to

$$(rG_r)'/r + imG_\phi/r + iaG_x = -\partial F_x/\partial X. \tag{15}$$

A straightforward elimination procedure then yields

$$LG_p = -T, \tag{16}$$

where L is the differential operator on the left-hand side of (9) and the linear functional T is defined by

$$\begin{aligned} T = A(X) \Bigg\{ & U_r \left[\frac{im}{r^2} (rF_\phi^0)' + iaF_x^{0\prime} \right] + (U_r F_r^0)'' \\ & + \left(\frac{1}{r} - \frac{2U_x'}{U_x - c} \right) (U_r F_r^0)' + ia \left(\frac{\partial F_p^0}{\partial X} + c\, \frac{\partial F_x^0}{\partial X} + F_x^0 \frac{\partial U_x}{\partial X} \right) \\ & - iU_x \frac{\partial}{\partial X} (aF_x^0) + U_x' \left(1 - \frac{2U_x}{U_x - c} \right) \frac{\partial F_r^0}{\partial X} \Bigg\} \\ & + 2A'(X) \left[ia(c - U_x)\, F_x^0 - \frac{U_x U_x'}{U_x - c} F_r^0 \right]. \end{aligned} \tag{17}$$

According to theorems for inhomogeneous eigenvalue problems, (16) has a solution satisfying the boundary conditions if and only if

$$\int_0^\infty \tilde{F}_p^0 \, T \, dr = 0. \tag{18}$$

By an inspection of (17) this condition yields an equation for the slowly varying amplitude $A(X)$:

$$k(X)\, dA/dX + l(X)\, A(X) = 0. \tag{19}$$

In the following we use the abbreviation $V = 1/(U_x - c)$. Now, by putting $l = l_1 + l_2$ and using (5)–(7) and (9), we can write

$$\begin{aligned} l_1(X) = -\frac{i}{a} \int_0^\infty & dr\, rF_p^{0\prime}\, V^3 \{ [U_r' + (VU_x' - 1/r)\, U_r]\, F_p^{0\prime} \\ & + U_r [2(m^2/r^2 + a^2) + V(U_x'' - U_x'/r)]\, F_p^0 \} \end{aligned} \tag{20}$$

and

$$l_2(X) = i\int_0^\infty dr\,\tilde{F}_p^0\left\{2a(D_X F_p^0 - VF_p^0 D_X U_x) + (F_p^0 + 2cF_p^{0\prime} U_x U_x' V^3/a^2) D_X a + \frac{V^2}{a}\left[\frac{F_p^{0\prime}}{V} D_X U_x' - 2cU_x' D_X F_p^{0\prime}\right] + \frac{2}{a} V^3 U_x' F_p^{0\prime}(2c - U_x) D_X U_x\right\}, \qquad (21)$$

where $D_X = \partial/\partial X$. Note that l_1 describes the influence of the transverse velocity U_r, whereas l_2 describes the effect of the streamwise variation. Finally, the coefficient of $A'(X)$ in (19) is given by

$$k(X) = 2i\int_0^\infty \tilde{F}_p^0[aF_p^0 - cU_x' V^2 F_p^{0\prime}/a]\,dr. \qquad (22)$$

Returning to (19) and considering (8), we may write

$$\mathbf{H} = A_0\mathbf{F}^0(X,r)\exp\left\{\int_{X_0}^X\left[\frac{i}{\epsilon}a(X') - l(X')/k(X')\right]dX' + i(m\phi - \beta t)\right\}, \qquad (23)$$

where A_0 is an arbitrary constant of integration. Now it is easy to see that each flow quantity (velocity, pressure, etc.) has its own growth rate. Furthermore, we may define local growth rates and phase speeds in the manner described by Crighton & Gaster, and in the appendix we show how to derive from (20)–(23) the corresponding formulae for axisymmetric disturbances ($m = 0$), which are also obtained by Crighton & Gaster.

For the mean velocity profile we use a form which was established by Michalke and later extended to the case of slowly diverging jet by Crighton & Gaster. This is given by

$$U_x = 0{\cdot}5\left\{1 + \tanh\left[b(x)\left(\frac{R}{r} - \frac{r}{R}\right)\right]\right\}, \qquad (24)$$

with

$$b(x) = 25(3x/R + 4)^{-1}, \qquad (25)$$

whereby the momentum thickness θ is given by $\theta(x)/R = 1/4b(x)$. If we assume that the rate of divergence is characterized by θ we can write

$$\epsilon = d\theta/dx = 0{\cdot}03. \qquad (26)$$

If we calculate the transverse mean velocity $V_r = \epsilon U_r$ by means of numerical integration of (24) and if we put $\epsilon D_X = \partial/\partial x$, we may reformulate (23) and get by elimination of the formal expansion parameter ϵ

$$\mathbf{H} = A_0\mathbf{F}^0(x,r)\exp\left\{\int_{x_0}^x [ia(x') - l(x')/k(x')]\,dx' + i(m\phi - \beta t)\right\}, \qquad (27)$$

whereby U_r must be replaced by V_r in (20). Seemingly, the second term in the exponential function in (27) plays the role of a small correction, and its neglect reduces (27) to a form proposed by Chan (1977). Since the magnitude of this term depends on the normalization of the eigenfunction, its omission is in general not justified.

3. Numerical results and discussion

We used a Runge–Kutta scheme and a modified Simpson routine. For the radial integrations, a minimum step width of $h = 0{\cdot}005R$ was found to ensure a sufficient degree of accuracy. The initial point x_0 of the axial integrations was arbitrarily chosen

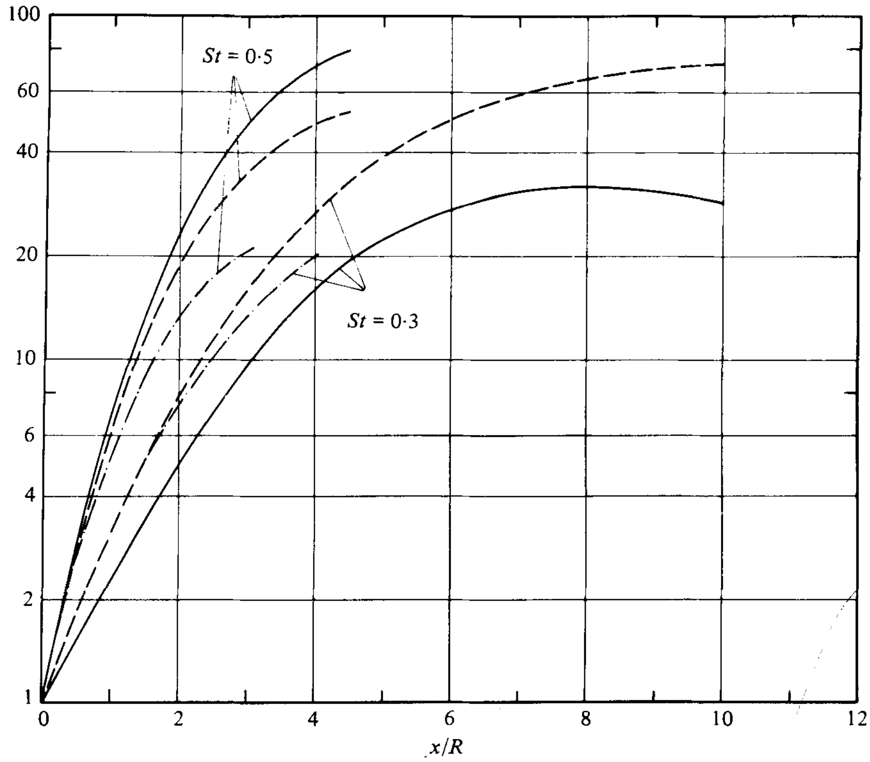

FIGURE 1. Gain pressure at centre of boundary layer.
——, $m = 0$; – – –, $m = 1$; –·–, $m = 2$.

to be $x_0 = 0$ and in order to eliminate the constant A_0 [cf. (23)]; the growth rates are referred to their values at $x = 0$. Furthermore, because of remarkable numerical difficulties, we stopped the integrations at axial positions where the inviscid theory still allows the determination of the eigenvalues.

In earlier measurements by Chan (1974) it was found that for higher values of the Strouhal number the disturbances are confined to the first part of the jet near the nozzle and that the Strouhal number of the maximally amplified modes is different for different radial locations. In the centre of the jet Chan reported a value of $St = 0{\cdot}35$, while on the centre-line of the boundary layer ($r = R$) the preferred Strouhal number is about 0·5.

These experimental findings are in close agreement with our theoretical prediction for the pressure gain in the centre of the boundary layer plotted in figure 1. From this plot we can see that modes with higher Strouhal numbers suffer a higher gain and that these modes decay under the influence of the viscosity at axial positions where modes with lower St are still growing. As regards the influence of the helicity number, we find that for $St = 0{\cdot}5$ the axisymmetric mode is more amplified than the first spiral mode. This result is consistent with the theory and the experiments by Chan (1977, figure 9), where the maximal gains for the axisymmetric mode and the first two spiral disturbances are found to be in the neighbourhood of the axial locations $x/R = 4$, $x/R = 3$ and $x/R = 2$, respectively. A comparison of our numerical data with values roughly extrapolated from the curves of Chan (1977) indicates that

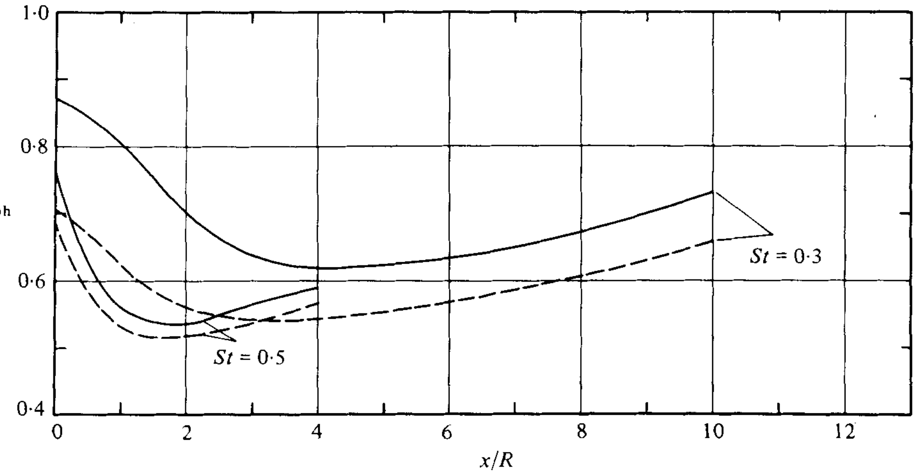

FIGURE 2. Phase speeds of pressure waves in the centre of the boundary layer. ——, $m = 0$; ---, $m = 1$.

our results overshoot the experimental ones of Chan by a factor of the order of 20 % at the position $x/R = 3$. This discrepancy might be explained by the comparatively high forcing level in Chan's experiments.

If we consider the pressure gain of modes excited at lower Strouhal numbers ($St = 0{\cdot}3$), we find that the first spiral mode suffers a greater pressure gain. This becomes more marked at positions further downstream, where the pressure gain of the axisymmetric mode reaches its maximum while the first helical mode continues to grow. This result is in good agreement with experimental measurements of co-spectral densities of pressure fluctuations in the centre of the boundary layer performed by Armstrong (1977). There the preferred Strouhal number was shown to take the value $St = 0{\cdot}45$, and for this particular Strouhal number the axisymmetric mode is the dominant disturbance, while for all other values of St the first spiral mode was found to be the most excited instability.

Finally, we may state that within the range of Strouhal numbers of interest to experimentalists both the axisymmetric mode and the first helical mode suffer a pressure gain of equal order of magnitude. Since the Strouhal number $St = 0{\cdot}3$ is supposed to be the preferred one at the centre-line, we may expect that the transverse components of the first spiral mode and the pressure and the axial component of the axisymmetric disturbance are dominant there. While in an intermediate regime ($r \simeq \frac{1}{2}R$) both disturbances are of equal importance, we suppose that, because $St \simeq 0{\cdot}5$ is the expected value of the Strouhal number in the centre of the boundary layer, the axisymmetric modes become the most amplified disturbances in a region around $r = R$.

The phase velocity of the pressure waves is defined by

$$c_{\text{ph}}(x,r) = \beta \Big/ \left[\mathscr{R}(a) - \mathscr{I}(l/k) + \frac{\partial}{\partial x} \arctan \frac{\mathscr{T}(F_p^0)}{\mathscr{R}(F_p^0)} \right],$$

and is plotted in figure 2 for the case $r = R$. From this we may see that, in agreement with experiments by Chan (1974), the phase velocity decreases with increasing Strouhal number. Furthermore, in analogy with the results of the local parallel theory (Michalke 1971) and in close resemblance to the trends found in experiments by Chan (1977), the helical modes (the phase velocities of the mode $m = 2$ are slightly below the ones of the mode with $m = 1$) exhibit essentially lower values of the phase velocity.

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Appendix

In the case of axisymmetric disturbances we may describe the quantities of the local parallel theory in terms of a stream function $\phi^0(x, r)$. Then the pressure is defined by

$$F_p^0 = -[(U_x - c)\,\phi^{0\prime} - U_x'\phi^0]/r, \tag{28}$$

and instead of (9) the disturbance equation is

$$(U_x - c)\,(d^2 - a^2)\,\phi^0 - (d^2 U_x)\,\phi^0 = 0, \tag{29}$$

where $d^2 = d^2/dr^2 - r^{-1}d/dr$. We now show the consistency of our theoretical results for the case of axisymmetric modes with the theory of Crighton & Gaster with respect to the function $k(x)$, only. The transformation of the function $l(x)$ follows the same lines. Introducing (28) into (22), we may write

$$k(x) = 2ia \int_0^\infty \frac{dr}{r}\,[\phi^{0\prime 2} - (2U_x - c)\,U_x'\,V^2\phi^{0\prime}\phi^0 + U_x\,U_x'^2\,V^3\phi^{02}]. \tag{30}$$

With the aid of a partial integration and by using (29), we can transform the first term on the right-hand side of (30). Equivalently, partial integration of the second term yields

$$k(x) = -ia \int_0^\infty dr\,\tilde{\phi}^0[2a^2(U_x - c) - c(d^2U_x)/(U_x - c)]\,\phi^0 \tag{31}$$

($\tilde{\phi}^0 = \phi^0/[r(U_x - c)]$). Apart from the pre-multiplier on the right-hand side, this is the formula presented by Crighton & Gaster (cf. equations (2.18), (2.23*a*) and (2.13) in their paper). Since this premultiplier will appear in the function $l(x)$ too, its influence will cancel.

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